Relativity and Cosmology I

Exam Solutions - January 2025

1. Parallel Transport and Curvature

(a) We can use the general formula for the Christoffel symbols:

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}) \tag{1}$$

or the quicker formulas for a diagonal metric (derived in problem set 5):

$$\begin{aligned} &\Gamma^{\bar\lambda}_{\bar\mu\bar\nu} = 0\,, && \Gamma^{\bar\lambda}_{\bar\mu\bar\mu} = -\frac{1}{2} g^{\bar\lambda\bar\lambda} \partial_{\bar\lambda} g_{\bar\mu\bar\mu}\,, \\ &\Gamma^{\bar\lambda}_{\bar\lambda\bar\lambda} = \frac{1}{2} g^{\bar\lambda\bar\lambda} \partial_{\bar\lambda} g_{\bar\lambda\bar\lambda}\,, && \Gamma^{\bar\lambda}_{\bar\mu\bar\lambda} = \frac{1}{2} g^{\bar\lambda\bar\lambda} \partial_{\bar\mu} g_{\bar\lambda\bar\lambda} \end{aligned} \tag{2}$$

where $\bar\lambda \neq \bar\mu \neq \bar\nu$ and no Einstein summation is implied in all these equations. Using these formulas, we find that the non-vanishing Christoffel symbols are

$$\Gamma^{r}_{\phi\phi} = -f f'\,, \qquad \Gamma^{\phi}_{r\phi} = \frac{f'}{f}\,. \tag{3}$$

(b) A surface with metric g is smooth at a certain point if g reduces to the flat metric close to that point. Given that the flat metric in polar coordinates is

$$ds^2 = dr^2 + r^2 d\phi^2 \tag{4}$$

smoothness is achieved if we have $f(0) = 0$ and $f'(0) = 1$. Another way to see that is to compute the circumference of a circle of radius r_0. We parametrize a circumference as

$$x^\mu(\lambda) = \begin{pmatrix} r(\lambda) \\ \phi(\lambda) \end{pmatrix} = \begin{pmatrix} r_0 \\ \lambda \end{pmatrix}\,. \tag{5}$$

Then, we get

$$C(r_0) = \int_0^{2\pi} d\lambda \sqrt{g_{\mu\nu} \dot x^\mu \dot x^\nu} = 2\pi f(r_0)\,. \tag{6}$$

For a small circle, the answer should be the same as in flat space, $2\pi r_0$. That means that in particular $f(0) = 0$ and $f'(0) = 1$. If $f(0) \neq 0$ that would mean $r = 0$ is not a point since the length of arcs at $r = 0$ would be nonzero. If $f(0) = 0$ but $f'(0) \neq 1$, we would have that an infinitesimal circle of radius $r \ll 1$ centered at the origin would have circumference $C = 2\pi f'(0) r$: there would be a conical defect.

Extra comment: In fact, if want the surface to be perfectly smooth then the Taylor expansion of $f(r)$ should only contain odd powers of r. To see this, we can move to cartesian coordinates $x = r\cos\phi$ and $y = r\sin\phi$ and demand that the metric components are analytic functions of x and y (in particular, they should not contain $\sqrt{x^2+y^2}$). Explicitly, the metric can be written as

$$ds^2 = dx^2 + dy^2 + \frac{f^2(r) - r^2}{r^4}(ydx - xdy)^2 \,. \tag{7}$$

Therefore, $f^2(r) - r^2 = \sum_{n=2}^{\infty} a_n r^{2n}$ follows from analyticity of the metric in cartesian coordinates. This is equivalent to $f - r = \sum_{n=2}^{\infty} b_n r^{2n-1}$.

(c) To compute the components of the Ricci tensor we start from the formula of the Riemann tensor and contract the indices

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = \partial_\rho \Gamma^{\rho}_{\mu\nu} - \partial_\nu \Gamma^{\rho}_{\mu\rho} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\rho\mu} \,. \tag{8}$$

We find the non-vanishing components to be

$$R_{rr} = -\frac{f''}{f} \,, \qquad R_{\phi\phi} = -ff'' \,. \tag{9}$$

The Ricci scalar is thus

$$R = g^{\mu\nu}R_{\mu\nu} = R_{rr} + f^{-2}R_{\phi\phi} = -2\frac{f''}{f} \,. \tag{10}$$

(d) The initial conditions we are given for the vector are

$$V^r(\lambda = 0) = 1 \,, \qquad V^\phi(\lambda = 0) = 0 \,. \tag{11}$$

We parallel transport this vector on the curve (5) for $\lambda \in [0, 2\pi]$. That means its components satisfy the differential equations

$$\frac{dx^\mu}{d\lambda}\nabla_\mu V^\nu = 0 \,. \tag{12}$$

Explicitly,

$$\begin{cases} \partial_\lambda V^r - ff'V^\phi = 0 \\ \partial_\lambda V^\phi + \frac{f'}{f}V^r = 0 \end{cases} \tag{13}$$

To solve this system of equations, we take a derivative of the first one with respect to λ and we substitute $\partial_\lambda V^\phi$ from the second into the first. We obtain

$$\begin{cases} \partial_\lambda^2 V^r = -f'^2 V^r \\ V^\phi = -\frac{1}{ff'}\partial_\lambda V^r \end{cases} \tag{14}$$

The solution to the first equation is a harmonic oscillator with frequency f'. The initial condition on V^r fixes the coefficient of the cosine

$$V^r(\lambda) = \cos(f'\lambda) + c_1 \sin(f'\lambda) \,. \tag{15}$$

Then, using the other equation and the initial condition on V^ϕ we get

$$V^\phi(\lambda) = -\frac{1}{f}\sin(f'\lambda) \,, \tag{16}$$

implying that $c_1 = 0$. Overall, the components of the vector after one turn are

$$\begin{pmatrix} V^r(2\pi) \\ V^\phi(2\pi) \end{pmatrix} = \begin{pmatrix} \cos[2\pi f'(r_0)] \\ -\frac{1}{f(r_0)}\sin[2\pi f'(r_0)] \end{pmatrix} . \tag{17}$$

This answer is consistent with the flat space case, in which $f'(r_0) = 1$ and in which the vector comes back to itself after one full turn. Notice that the norm of the vector does not change as it rotates

$$g_{\mu\nu}(x(\lambda))V^r(\lambda)V^\phi(\lambda) = 1 . \tag{18}$$

Given this fact, the angle α the final vector makes with the initial vector satisfies the equation

$$\cos\alpha = g_{\mu\nu}(r_0, 0)V^\mu(0)V^\nu(2\pi) = \cos[2\pi f'(r_0)] \tag{19}$$

or in other words $\alpha = 2\pi f'(r_0)$.

Extra comment: It might be surprising to see that even in flat space the vector rotates before coming back to itself. The reason is that whether a vector rotates or not is a coordinate-dependent statement. In the setup we are given, if we think of it in cartesian coordinates, the vector always points in the x direction, its cartesian components never rotate as we parallel transport it in a circle. But now think of the same phenomenon in polar coordinates. The vector is still always pointing to the right, but that means that its radial component is rotating, it is not always pointing outwards from the center.

(e) To properly integrate the Ricci scalar over an area we need to use the correct volume element

$$\int_\Sigma d^2x\sqrt{g}R \equiv \int_0^{r_0} \sqrt{g}\; dr \int_0^{2\pi} d\phi R = 2\pi \int_0^{r_0} dr \left(-2\frac{f''}{f}\right) f = 4\pi(1 - f'(r_0)). \tag{20}$$

In terms of α, we can thus write

$$\alpha = -\frac{1}{2}\int_\Sigma d^2x\sqrt{g}R \qquad \mod 2\pi \tag{21}$$

where Σ is the region inside the circle. In fact, this is a particular case of the Gauss-Bonnet theorem.

2. Massive Scalar Field

(a) To compute the equations of motion we take a functional variation of the action with respect to $\phi(x)$

$$\delta S = -\frac{1}{2}\int d^4x\sqrt{-g}\left(2g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\delta\phi + 2m^2\phi\delta\phi\right) \tag{22}$$

Note that we replaced the partial derivatives by covariant derivatives since they act on scalars. This is convenient because now we can integrate by parts

$$\delta S = -\int d^4x\sqrt{-g}\left(-g^{\mu\nu}\nabla_\mu\nabla_\nu\phi + m^2\phi\right)\delta\phi. \tag{23}$$

Setting the variation to be zero for any $\delta\phi$ yields the equations of motion

$$\nabla^\mu \nabla_\mu \phi = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \partial^\mu \phi \right) = m^2 \phi. \tag{24}$$

Alternatively we can keep partial derivatives and integrate by parts in the usual way. We get

$$\begin{aligned} \delta S &= -\int d^4x \sqrt{-g} \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \delta\phi + m^2 \phi \delta\phi \right) \\ &= -\int d^4x \sqrt{-g} \left(-\frac{1}{\sqrt{-g}} \partial_\nu \sqrt{-g} g^{\mu\nu} \partial_\mu \phi + m^2 \phi \right) \delta\phi, \end{aligned} \tag{25}$$

and setting the integrant to zero gives the equations of motion above.

(b) Let us first compute

$$\nabla^\mu \nabla_\mu e^{imS(x)} = \nabla^\mu \left(e^{imS} \nabla_\mu S \right) = -m^2 e^{imS} (\nabla_\mu S)(\nabla^\mu S) + \mathcal{O}(m). \tag{26}$$

Plugging this in the equations of motion we get

$$\nabla_\mu S \ \nabla^\mu S + 1 = 0 \tag{27}$$

(c) We consider the curve with tangent vector

$$w^\mu = g^{\mu\nu} \partial_\nu S = \nabla^\mu S. \tag{28}$$

To show that this is a geodesic we consider the parallel transport equation

$$0 = w^\mu \nabla_\mu w^\nu = w^\mu \nabla_\mu \nabla^\nu S = (\nabla^\mu S) \nabla^\nu (\nabla_\mu S). \tag{29}$$

Notice that $\nabla_\nu \nabla_\mu S = \nabla_\mu \nabla_\nu S$ because S is a scalar. You can check this explicitly using $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$. Taking the covariant derivative ∇^ν of the equations of motion for S we get

$$2 \nabla^\mu S \nabla^\nu \nabla_\mu S = 0 \tag{30}$$

which then implies the geodesic equation.

Physically the derivative of the phase $\nabla_\mu S = w_\mu$ gives an orthogonal vector to the wavefront of the wave described by $\phi(x)$. We can see this by looking at plane waves $\phi(x) \sim e^{ik_\mu x^\mu}$. If we quantize the fields, the quanta are particles with 4-velocity w^μ, and we just derived that (in the limit of large mass) those particles travel on geodesics. The parameter $\lambda \equiv \tau$ is the proper time of the particles.

3. Einstein Ring

(a) The geodesic equations follow from the Euler-Lagrange equations with the Lagrangian

$$\mathcal{L}(x^\mu, \dot{x}^\mu) = g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu, \qquad \frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0, \tag{31}$$

where a dot means a derivative with respect to the affine parameter λ. If the Lagrangian is independent of one of the coordinates, then we get a conserved a quantity

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = 0 \implies \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = 0 \implies \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = cste. \tag{32}$$

The coordinates x^μ can only appear in the Lagrangian through the metric $g_{\mu\nu}$. Hence if the metric in independent of one of the coordinates, we get conserved quantities. The metric

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)dr^2 + r^2(1-2\Phi)d\theta^2 + r^2(1-2\Phi)\sin^2\theta d\phi^2 \tag{33}$$

does not depend on t and ϕ. Furthermore we can set without loss of generality $\theta = \pi/2$. We then get the conserved quantities

$$E = (1+2\Phi)\dot{t}, \qquad L = (1-2\Phi)r^2\dot{\phi}. \tag{34}$$

A physical interpretation of those conserved quantities comes from looking in the asymptotic region $r \to \infty$ where the geometry becomes Minkowski spacetime and $E = \dot{t} = p^0$ is the energy of the particle while $L = r^2\dot{\phi} = p_\phi$ is its angular momentum.

We are interested in the motion of photons that travel on null geodesics which means

$$0 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -(1+2\Phi)\dot{t}^2 + (1-2\Phi)\dot{r}^2 + r^2(1-2\Phi)\dot{\phi}^2. \tag{35}$$

We can eliminate the dependence on $\dot{t}$ and $\dot{\phi}$ by using the conserved quantities. We get

$$0 = -\frac{E^2}{1+2\Phi} + (1-2\Phi)\dot{r}^2 + \frac{L^2}{r^2(1-2\Phi)}. \tag{36}$$

To express things in terms of $x(\phi)$ we use the chain rule

$$\frac{dx}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\phi}} = -\dot{r}\frac{1-2\Phi}{L}. \tag{37}$$

We therefore get

$$0 = -\frac{1-2\Phi}{1+2\Phi}\frac{E^2}{L^2} + (\frac{dx}{d\phi})^2 + x^2. \tag{38}$$

Expanding in small Φ and using $\Phi = -GMx$ we get

$$V(x) = x^2 - \frac{E^2}{L^2}(1+4GMx). \tag{39}$$

Setting $x(\phi) = \sin(\phi)/b$ and $\Phi = 0$, we get

$$b = \frac{L}{E}, \tag{40}$$

which is consistent with the explanation in the next question.

(b) First let us relate the conserved quantities E and L to the impact parameter b. As explained before, in the asymptotic region the conserved quantities E and L are respectively the energy and angular momentum of the photon. The angular momentum can also be computed as

$$\vec{L} = \vec{r} \wedge \vec{p} = rp\sin\alpha\vec{e}_z. \tag{41}$$

The spatial momentum of the photon p is simply related to its energy by $p = E$, and usual trigonometric relations yield $b = r \sin\alpha$, where b is the impact parameter. Therefore we get

$$L = bE \Leftrightarrow b = \frac{L}{E}. \tag{42}$$

Expanding the equations of motion around the straight trajectory $x(\phi) = \frac{\sin\phi}{b} + \epsilon\frac{y}{b}$, with $\epsilon = 2GM/b \ll 1$, being the small parameter, we get the linearized equation of motion

$$2\cos\phi\frac{dy}{d\phi} + 2\sin\phi\; y - 2\sin\phi = 0 + \mathcal{O}(\epsilon). \tag{43}$$

To solve this differential equation we use the variable

$$u = y - 1, \tag{44}$$

and the differential equation becomes

$$\frac{du}{d\phi} + \tan\phi\; u = 0. \tag{45}$$

We can solve it using the separation of variables. We get

$$\int_{u_0}^{u} \frac{du}{u} = -\int_0^{\phi} d\phi \tan\phi = \int_{-1}^{\cos\phi} d\cos\phi \frac{1}{\cos\phi} \tag{46}$$

The solution is therefore

$$\frac{u}{u_0} = \cos\phi \tag{47}$$

The integration constant u_0 follows from considering a trajectory incoming from infinity at $\phi = 0$. Hence $y(0) = 0$ which implies $u(0) = u_0 = -1$. We therefore get

$$y(\phi) = 1 - \cos\phi. \tag{48}$$

To relate y to the deflection angle α we use

$$\tan\alpha = \frac{b}{D} = \frac{b}{r(\phi = \pi)} = bx(\pi) = \epsilon y(\pi). \tag{49}$$

Therefore in the small angle approximation we have

$$\alpha = \epsilon y(\pi) = \frac{4GM}{b}. \tag{50}$$

Alternatively, we can define the deflection angle as the angle at which the photon goes to infinity. More precisely, the deflection angle α is defined by

$$0 = x(\pi + \alpha) = \frac{1}{b}\left(-\sin\alpha + \epsilon(1 + \cos\alpha)\right) = \frac{1}{b}\left(2\epsilon - \alpha + O(\alpha^2)\right)\,. \tag{51}$$

(c) In the approximation we are considering, in which S is so far away that its lightrays arrive horizontally at the lens L, the Einstein angle θ_E and the deflection angle α are the same. Then, what we just found is that

$$\theta_E = \frac{4GM}{b}\,. \tag{52}$$

At the same time, from the geometry of the problem it is clear that

$$\frac{b}{D} = \tan\theta_E \sim \theta_E \, . \tag{53}$$

We can thus eliminate b and obtain

$$\theta_E = 2\sqrt{\frac{GM}{D}} \, . \tag{54}$$

(d) Restoring units and inverting the equation (54) we get

$$M = \frac{\theta_E^2 c^2 D}{4G} \, . \tag{55}$$

Recalling the definitions of arcsecond

$$2\pi = 60 \times 60 \times 360 \text{ arcsec} \tag{56}$$

and lightyear

$$1\text{ly} = c \times 365 \times 24 \times 3600 \tag{57}$$

we get

$$M \approx 5 \times 10^{12} M_\odot \tag{58}$$